

INJECTIVE AND PROJECTIVE HEYTING ALGEBRAS⁽¹⁾

BY

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The determination of the injective and projective members of a category is usually a challenging problem and adds to knowledge of the category. In this paper we consider these questions for the category of Heyting algebras. There has been a lack of uniformity in terminology in recent years. In [6] Heyting algebras are referred to as pseudo-Boolean algebras, and in [1] they are called Brouwerian lattices. We would argue for retaining the name Heyting algebras for the reasons given in [2, p. 162], reserving the name Brouwerian algebras for the algebras dual to Heyting algebras. The fundamental paper on Brouwerian algebras (and therefore Heyting algebras) is [4].

We shall show that a Heyting algebra is injective if and only if it is a complete Boolean algebra. The determination of projective Heyting algebras is, as usual, more difficult. We shall characterize all projective Heyting algebras which are finite or are chains. Some other results on projective algebras are given.

1. Definitions and preliminaries. If S is a subset of a lattice L , then $\Sigma(S)$ denotes the least upper bound in L of S , and $\Pi(S)$ denotes the greatest lower bound of S . If $S = \{x, y\}$, then $\Sigma(S)$ and $\Pi(S)$ are written $x + y$ and xy . The largest and smallest elements of a lattice L are denoted by 1 (or 1_L) and 0 respectively.

A Heyting algebra is a lattice with 0 such that for any elements a, b there is an element $a \rightarrow b$ such that $ax \leq b$ if and only if $x \leq a \rightarrow b$. We use $\neg a$ to denote $a \rightarrow 0$. Every Heyting algebra is a distributive lattice. The lattice of all open sets of a topological space is a Heyting algebra. We state without proof some elementary properties of Heyting algebras.

THEOREM 1.1. *If a, b , and c are any elements of a Heyting algebra, then the following hold:*

- (1) $a \leq b$ if and only if $a \rightarrow b = 1$.
- (2) $1 \rightarrow a = a$.
- (3) $a \rightarrow b \geq b$.
- (4) If $b \leq c$, then $a \rightarrow b \leq a \rightarrow c$ and $c \rightarrow a \leq b \rightarrow a$.

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- (5) $a \rightarrow (b \rightarrow c) = ab \rightarrow c$.
- (6) If $a \leq c$ and $a \rightarrow b = b$, then $c \rightarrow b = b$.
- (7) $(a + b) \rightarrow c = (a \rightarrow c)(b \rightarrow c)$.
- (8) $a \leq (a \rightarrow b) \rightarrow b$.
- (9) $((a \rightarrow b) \rightarrow b) \rightarrow b = a \rightarrow b$.
- (10) $((a \rightarrow b) \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$.
- (11) $(a \rightarrow b) \rightarrow ((a \rightarrow b) \rightarrow b) = (a \rightarrow b) \rightarrow b$.
- (12) If $a \leq b$, $a(b \rightarrow c) = ac$.

From the point of view of universal algebra, Heyting algebras are regarded as algebras $\langle H, +, \cdot, \rightarrow, 0 \rangle$ with three binary operations and a distinguished element. Accordingly subalgebras and homomorphisms of Heyting algebras are subsets and functions which preserve the operations and 0. A Heyting homomorphism f is one-to-one (that is, f is a monomorphism) if and only if $f(u) = 1$ only for $u = 1$. We shall usually refer to Heyting homomorphisms by using the word homomorphism without qualification. A Heyting algebra H is said to be injective (in the category of Heyting algebras) if for any Heyting monomorphism $f: B \rightarrow A$ and any homomorphism $h: B \rightarrow H$, there exists a homomorphism $g: A \rightarrow H$ such that $gf = h$. A Heyting algebra H is called projective if for any Heyting epimorphism $f: A \rightarrow B$ and any homomorphism $h: H \rightarrow B$, there exists a homomorphism $g: H \rightarrow A$ such that $fg = h$.

The identity map of a set A is denoted by I_A . A Heyting algebra H is a retract of an algebra K if there exist homomorphisms $f: K \rightarrow H$ and $g: H \rightarrow K$ such that $fg = I_H$. A retract of a projective Heyting algebra is projective. A Heyting algebra is projective if and only if it is a retract of a free Heyting algebra. If $f: K \rightarrow H$ is a Heyting epimorphism and H is projective, then H is a retract of K .

Boolean algebras will be regarded as algebras $\langle B, +, \cdot, \bar{} \rangle$ where $\bar{a}$ denotes the complement of a . There are certain connections between Heyting algebras and Boolean algebras which we shall use. If $\langle B, +, \cdot, \bar{} \rangle$ is a Boolean algebra, then $\langle B, +, \cdot, \rightarrow, 0_B \rangle$ is a Heyting algebra, where $a \rightarrow b = \bar{a} + b$. If B_1 and B_2 are Boolean algebras then a map $f: B_1 \rightarrow B_2$ is a Heyting homomorphism if and only if it is a Boolean homomorphism. If H is a Heyting algebra, let $R(H)$ be the set of regular elements, that is, $R(H) = \{x \in H : x = \neg \neg x\}$. The following theorem is well known.

THEOREM 1.2. *If H is a Heyting algebra, then $\langle R(H), +, \cdot, \neg \rangle$ is a Boolean algebra, where $a + b = \neg \neg (a + b)$. The canonical map $\varphi: H \rightarrow R(H)$ defined by $\varphi(x) = \neg \neg x$ is a Heyting epimorphism.*

If H is a Heyting algebra, then $\langle H, +, \cdot, \neg \rangle$ is a Boolean algebra if and only if $\neg x = 0$ only for $x = 1$. This follows immediately from the identity $\neg(x + \neg x) = 0$.

A linearly ordered set with 0 and 1 will be called a *chain*. Every chain is a Heyting algebra in which $x \rightarrow y = y$ for $x > y$ and $x \rightarrow y = 1$ for $x \leq y$. By a subchain of a Heyting algebra, we mean a subalgebra which is a chain.

If L is a lattice and $x \in L$, let $L_{(x)}$ denote $\{y \in L : y \leq x\}$ and $L^{(x)}$ denote $\{y \in L : y \geq x\}$.

2. Injective Heyting algebras.

LEMMA 2.1. *Let $f: H \rightarrow K$ and $g: K \rightarrow H$ be order-preserving maps of partially ordered sets such that $gf = I_H$. If K is complete, then H is complete.*

Proof. Let $\{x_\alpha : \alpha \in I\}$ be any family of elements of H . Let $y = \Sigma\{f(x_\alpha) : \alpha \in I\}$. Then $g(y) \geq gf(x_\alpha) = x_\alpha$ for all $\alpha \in I$. If $z \in H$ and $z \geq x_\alpha$ for all α , then $f(z) \geq f(x_\alpha)$ for all α . Therefore $f(z) \geq y$ and $z = gf(z) \geq g(y)$. Hence $g(y) = \Sigma\{x_\alpha : \alpha \in I\}$.

THEOREM 2.2. *If H is an injective Heyting algebra, then H is a complete Boolean algebra.*

Proof. Let K be the set of all ideals of H . Then K is a complete Heyting algebra [1, p. 129]. For $x \in H$, let $f(x) = H_{(x)}$. Then $f: H \rightarrow K$ is a monomorphism. Since H is injective, there exists a homomorphism $g: K \rightarrow H$ such that $gf = I_H$. By 2.1, H is complete.

To prove H is a Boolean algebra, we must show $\neg x = 0$ only for $x = 1$. Suppose there exists $a \in H$ such that $\neg a = 0$ and $a < 1$. Then $\{0, a, 1\}$ is a subchain of H . Extend it to a maximal subchain C of H . Let $g: C \rightarrow H$ be defined by $g(x) = x$. Let D be the chain obtained by adding to C a new element α such that $0 < \alpha < x$ for all $x \in C - \{0\}$. Then C is a subchain of D . Since H is injective, there exists a homomorphism $h: D \rightarrow H$ which extends g . Now $h(D)$ is a subchain of H containing C . By the maximality of C , $h(\alpha) \in C$. But $h(\alpha) \neq 0$, since $\neg h(\alpha) = h(\neg \alpha) = h(0) = 0$. Also $h(\alpha) \leq h(a) = a < 1$. If $x \in C - \{0\}$, then $h(\alpha) \leq x$ and $x \rightarrow h(\alpha) = h(x \rightarrow \alpha) = h(\alpha) < 1$. Therefore $h(\alpha) < x$ for all $x \in C - \{0\}$ by 1.1 (1). Thus $h(\alpha) \notin C$ and we have a contradiction.

LEMMA 2.3. *Let $h: H \rightarrow B$ be a Heyting homomorphism from a Heyting algebra H to a Boolean algebra B . Then there exists a Boolean homomorphism $h_1: R(H) \rightarrow B$ such that $h_1\varphi = h$, where $\varphi: H \rightarrow R(H)$ is the canonical map of 1.2.*

Proof. Since $h(\neg x) = \overline{h(x)}$, we have $h(\neg \neg x) = h(x)$ for all $x \in H$. Let h_1 be the restriction of h to $R(H)$. If $x, y \in R(H)$, $h(x \dot{+} y) = h(\neg \neg (x + y)) = h(x + y) = h_1(x) + h_1(y)$. Therefore h_1 is a Boolean homomorphism. Also $h_1\varphi(x) = h(\neg \neg x) = h(x)$ for all $x \in H$.

LEMMA 2.4. *Let $f: H_1 \rightarrow H_2$ be a Heyting homomorphism. Then there exists a Boolean homomorphism $f': R(H_1) \rightarrow R(H_2)$ such that $\varphi_2 f = f' \varphi_1$, where φ_1, φ_2 are the canonical maps of 1.2. If f is one-to-one, so is f' .*

Proof. The existence of f' follows immediately from 2.3. Assume f is one-to-one. Suppose $x \in R(H_1)$ and $f'(x) = 1$. Then $1 = f'(x) = f'(\neg \neg x) = f' \varphi_1(x) = \varphi_2 f(x) = \neg \neg f(x) = f(\neg \neg x)$. Therefore $\neg \neg x = 1$ and so $x = 1$. Hence f' is one-to-one.

THEOREM 2.5. *A Heyting algebra H is injective if and only if it is a complete Boolean algebra.*

Proof. The necessity was proved in 2.2. Suppose H is a complete Boolean algebra. Let $f: H_1 \rightarrow H_2$ be a Heyting monomorphism, and let $h: H_1 \rightarrow H$ be a Heyting homomorphism. Let $\varphi_i: H_i \rightarrow R(H_i)$ be the canonical maps, $i=1, 2$. By 2.3 there exists a Boolean homomorphism $h_1: R(H_1) \rightarrow H$ such that $h_1\varphi_1=h$. By 2.4 there exists a Boolean monomorphism $f': R(H_1) \rightarrow R(H_2)$ such that $\varphi_2f=f'\varphi_1$. Now Sikorski's theorem implies that H is injective in the category of Boolean algebras [3, p. 141]. Therefore there exists a Boolean homomorphism $g_1: R(H_2) \rightarrow H$ such that $g_1f'=h_1$. Let $g=g_1\varphi_2: H_2 \rightarrow H$. Then $gf=g_1\varphi_2f=g_1f'\varphi_1=h_1\varphi_1=h$.

3. Star sums of Heyting algebras.

DEFINITION 3.1. If L is a lattice, then $L \oplus 1$ denotes the lattice obtained by adding a new element 1 such that $1 > x$ for all $x \in L$.

DEFINITION 3.2. A *pre-Heyting algebra* is a lattice L such that $x \rightarrow y$ exists whenever $x, y \in L$ and $x \not\leq y$.

DEFINITION 3.3. A *pre-subalgebra* of a pre-Heyting algebra L is a sublattice S such that $x \rightarrow y \in S$ whenever $x, y \in S$ and $x \not\leq y$.

If L is a pre-Heyting algebra with 0 and 1, then L is a Heyting algebra by 1.1 (1). If L is a pre-Heyting algebra with 0, then $L \oplus 1$ is a Heyting algebra and L is a pre-subalgebra of $L \oplus 1$.

DEFINITION 3.4. If L_1 is a pre-Heyting algebra and L_2 is a Heyting algebra, a function $f: L_1 \rightarrow L_2$ is called a *pre-homomorphism* if f is a lattice homomorphism and $f(x \rightarrow y) = f(x) \rightarrow f(y)$ whenever $x \not\leq y$.

DEFINITION 3.5. An element α of a lattice L is called a *node* if α is comparable with every element of L .

THEOREM 3.6. *If α is an element of a Heyting algebra H , then $\langle H^{(\alpha)}, +, \cdot, \rightarrow, \alpha \rangle$ is a Heyting algebra. If α is a node, then $H_{(\alpha)}$ is a Heyting algebra which is a pre-subalgebra of H .*

Proof. The proof is easy and is omitted.

DEFINITION 3.7. Suppose δ is an ordinal such that $0 < \delta \leq \omega$, and H_n is a Heyting algebra for each n , $0 \leq n < \delta$. By $\Sigma_{n < \delta}^* H_n$ we mean a lattice H with nodes $0_H = \alpha_0 < \alpha_1 < \dots < \alpha_\delta = 1_H$ such that for each n , H_n is isomorphic with

$$H'_n = \{x \in H: \alpha_n \leq x \leq \alpha_{n+1}\}.$$

We will usually identify H_n with H'_n , so that $\alpha_n = 0_{H_n}$ and $\alpha_{n+1} = 1_{H_n}$. If δ is finite, then $\Sigma_{n < \delta}^* H_n$ is written $H_0 * H_1 * \dots * H_{\delta-1}$.

THEOREM 3.8. *If $H = \Sigma_{n < \delta}^* H_n$, then H is a pre-Heyting algebra, and for each n , H_n is a pre-subalgebra of H . If δ is finite, H is a Heyting algebra.*

Proof. Suppose $x \in H_n$, $y \in H_m$ and $x \not\leq y$. It is easy to verify that if $m=n$, then $x \rightarrow y$ has the same meaning in H as in H_n , while if $m < n$ and $y < 0_{H_n}$ then $x \rightarrow y = y$.

LEMMA 3.9. Suppose $f_0: H_0 \rightarrow K_0$ is a Heyting isomorphism and $f_1: H_1 \rightarrow K_1$ is a Heyting homomorphism. Then the function $g: H_0 * H_1 \rightarrow K_0 * K_1$ defined by $g(x) = f_i(x)$ for $x \in H_i$, $i = 1, 2$, is a Heyting homomorphism.

Proof. This follows easily from 3.8 and its proof.

DEFINITION 3.10. Let B_2 and B_4 denote the Boolean algebras with two and four elements respectively.

Observe that if H is a Heyting algebra, then $H \oplus 1 = H * B_2$.

LEMMA 3.11. Let H be a Heyting algebra with a node α such that $H^{(\alpha)} = \{\alpha, 1_H\}$, and let F be a Heyting algebra. Suppose $g: H_{(\alpha)} \rightarrow F$ is a pre-homomorphism, and x is any member of F . Let $h: H \rightarrow F$ be the extension of g such that $h(1_H) = x + (x \rightarrow g(\alpha))$. Then h is a pre-homomorphism.

Proof. h is obviously a lattice homomorphism, since $h(1_H) \geq g(\alpha)$ by 1.1(3). If $u \in H_{(\alpha)}$ we have

$$\begin{aligned} h(1_H) \rightarrow h(u) &= (x + (x \rightarrow g(\alpha))) \rightarrow g(u) \\ &= (x \rightarrow g(u)) \cdot ((x \rightarrow g(\alpha)) \rightarrow g(u)) \quad \text{by 1.1(7)} \\ &= (x \rightarrow g(u)) \cdot g(u) \quad \text{by 1.1(4) and (12)} \\ &= g(u) \quad \text{by 1.1(3)} \\ &= h(u) = h(1_H \rightarrow u). \end{aligned}$$

LEMMA 3.12. Let H be a Heyting algebra with a node α such that $H^{(\alpha)} = \{\alpha, \beta, \gamma, 1_H\}$ is isomorphic with B_4 . Let F be a Heyting algebra, let $g: H_{(\alpha)} \rightarrow F$ be a pre-homomorphism, and let y be any member of F . Let $h: H \rightarrow F$ be the extension of g such that $h(\beta) = (y \rightarrow g(\alpha)) \rightarrow g(\alpha)$, $h(\gamma) = y \rightarrow g(\alpha)$, and $h(1_H) = h(\beta) + h(\gamma)$. Then h is a pre-homomorphism.

Proof. Since $h(\beta)$, $h(\gamma)$ and $h(1_H)$ are all $\geq g(\alpha)$, and $h(\beta)h(\gamma) = g(\alpha)$ by 1.1(12) and (3), it is easily seen that h is a lattice homomorphism. Note that $\beta \rightarrow \alpha = \beta \rightarrow \gamma = \gamma$ and $\gamma \rightarrow \alpha = \gamma \rightarrow \beta = \beta$.

If $u < \alpha$, then $h(\beta) \rightarrow h(u) = h(\beta) \rightarrow g(u) = g(u)$, by 1.1(6), since $h(\beta) \geq g(\alpha)$ and $g(\alpha) \rightarrow g(u) = g(\alpha \rightarrow u) = g(u)$ by 3.6. Therefore $h(\beta) \rightarrow h(u) = h(u) = h(\beta \rightarrow u)$ by 1.1(6). A similar argument shows $h(\gamma) \rightarrow h(u) = h(\gamma \rightarrow u)$. Also $h(\beta) \rightarrow h(\alpha) = h(\gamma)$ by 1.1(9), and so $h(\beta) \rightarrow h(\alpha) = h(\beta \rightarrow \alpha)$. Obviously $h(\gamma) \rightarrow h(\alpha) = h(\beta) = h(\gamma \rightarrow \alpha)$. Next $h(\beta) \rightarrow h(\gamma) = h(\gamma) = h(\beta \rightarrow \gamma)$ by 1.1(10), and $h(\gamma) \rightarrow h(\beta) = h(\beta) = h(\gamma \rightarrow \beta)$ by 1.1(11). Finally, by using 1.1(7), it is easily seen that $h(1_H) \rightarrow h(\alpha) = h(\alpha) = h(1_H \rightarrow \alpha)$, $h(1_H) \rightarrow h(u) = h(u) = h(1_H \rightarrow u)$ for $u < \alpha$, $h(1_H) \rightarrow h(\beta) = h(1_H \rightarrow \beta)$, and $h(1_H) \rightarrow h(\gamma) = h(1_H \rightarrow \gamma)$.

THEOREM 3.13. Let $H = (\Sigma_{n < \delta}^* H_n) \oplus 1$, where $0 < \delta \leq \omega$ and, for each n , H_n is isomorphic with B_2 or B_4 . Then H is a projective Heyting algebra.

Proof. Let $0_H = \alpha_0 < \alpha_1 < \dots < \alpha_\delta = 1_H$ be the nodes of H . Let

$$N = \{n < \delta : H_n \cong B_4\}.$$

For $n \in N$, $H_n = \{\alpha_n, \beta_n, \gamma_n, \alpha_{n+1}\}$, where β_n and γ_n are the atoms of H_n . For $n \notin N$, $H_n = \{\alpha_n, \alpha_{n+1}\}$. Let F be a free Heyting algebra with the set $\{x_n : 0 < n < \delta\} \cup \{y_n : n \in N\}$ of free generators. There exists a homomorphism $f: F \rightarrow H$ such that $f(x_n) = \alpha_n$ for $0 < n < \delta$ and $f(y_n) = \beta_n$ for $n \in N$. Clearly f is an epimorphism. We wish to show H is a retract of F . For each n , let $K_n = \bigcup_{i \leq n} H_i$ and $K_{-1} = \{0_H\}$. For $-1 \leq n < \delta$, define $g_n: K_n \rightarrow F$ inductively as follows.

Let $g_{-1}(0_H) = 0_F$. If $n \geq 0$, let g_n be the extension of g_{n-1} such that for $n \notin N$,

$$g_n(\alpha_{n+1}) = x_{n+1} + (x_{n+1} \rightarrow g_{n-1}(\alpha_n)),$$

and for $n \in N$,

$$g_n(\beta_n) = (y_n \rightarrow g_{n-1}(\alpha_n)) \rightarrow g_{n-1}(\alpha_n)$$

$$g_n(\gamma_n) = y_n \rightarrow g_{n-1}(\alpha_n) \quad \text{and}$$

$$g_n(\alpha_{n+1}) = g_n(\beta_n) + g_n(\gamma_n).$$

By 3.11 and 3.12, it follows by induction that g_n is a pre-homomorphism for $-1 \leq n < \delta$. Let g be the union (common extension) of all the g_n to $\Sigma_{n < \delta}^* H_n$. Then g is a pre-homomorphism. Extend g to all of H by defining $g(1_H) = 1_F$. Then $g: H \rightarrow F$ is obviously a homomorphism.

To show $fg = I_H$, first note that $fg_{-1}(0_H) = f(0_F) = 0_H$. Suppose $n \geq 0$, and we have shown $fg_{n-1} = I_{K_{n-1}}$. If $n \notin N$, then $fg_n(\alpha_{n+1}) = \alpha_{n+1} + (\alpha_{n+1} \rightarrow \alpha_n) = \alpha_{n+1}$. If $n \in N$, then

$$fg_n(\beta_n) = (\beta_n \rightarrow \alpha_n) \rightarrow \alpha_n = \gamma_n \rightarrow \alpha_n = \beta_n, \quad fg_n(\gamma_n) = \beta_n \rightarrow \alpha_n = \gamma_n,$$

and

$$fg_n(\alpha_{n+1}) = \beta_n + \gamma_n = \alpha_{n+1}.$$

Therefore $fg_n = I_{K_n}$ for all $n < \delta$. Since $fg(1_H) = 1_H$, it follows that $fg = I_H$. Thus H is a retract of a free Heyting algebra, and so H is projective.

4. Finite projective algebras.

LEMMA 4.1. *Suppose H_0 and H_1 are Heyting algebras and $H_0 * H_1$ is projective. If $f: A \rightarrow B$ is a Heyting epimorphism such that $f(x) = 0$ only for $x = 0$, and $h: H_1 \rightarrow B$ is a homomorphism, then there exists a homomorphism $g: H_1 \rightarrow A$ such that $fg = h$.*

Proof. Let $f': H_0 * A \rightarrow H_0 * B$ be defined by $f'(x) = x$ for $x \in H_0$ and $f'(x) = f(x)$ for $x \in A$. Let $h': H_0 * H_1 \rightarrow H_0 * B$ be defined similarly. By 3.9, f' and h' are homomorphisms and f' is an epimorphism. Since $H_0 * H_1$ is projective, there exists a homomorphism $g': H_0 * H_1 \rightarrow H_0 * A$ such that $f'g' = h'$. Since $f'g'(0_{H_1}) = h'(0_{H_1}) = 0_B$, we have $g'(0_{H_1}) = 0_A$. Hence $g'(x) \in A$ for all $x \in H_1$. Let g be the restriction of g' to H_1 . Then $g: H_1 \rightarrow A$ is a homomorphism. Clearly $fg(x) = h(x)$ for all $x \in H_1$.

LEMMA 4.2. *Let F be a filter in a Heyting algebra H . Then the following are equivalent:*

(1) *F is a maximal filter.*

(2) *The function $f: H \rightarrow B_2$ such that $f(x)=1$ if and only if $x \in F$ is a homomorphism.*

(3) *For each $x \in H$, exactly one of $x, \neg x$ is in F .*

If $a \in H$, then a is an atom if and only if $H^{(a)}$ is a maximal filter.

Proof. This is any easy consequence of [6, Chapter I, 13.10].

LEMMA 4.3. *Let $b_1, \dots, b_n$ be all the distinct atoms of a finite Boolean algebra B . Let $a_1, \dots, a_n$ be atoms (not necessarily distinct) of a Heyting algebra H . Then there exists a homomorphism $h: H \rightarrow B$ such that $h(x) = \sum \{b_i : a_i \leq x\}$ for all $x \in H$.*

Proof. By 4.2 there exist homomorphisms $g_i: H \rightarrow B_2$ such that $g_i(x)=1$ if and only if $x \geq a_i$. Let $g: H \rightarrow (B_2)^n$ be the homomorphism defined by $g(x) = (g_1(x), \dots, g_n(x))$. There exists an isomorphism $f: (B_2)^n \rightarrow B$ such that $f(e_i) = b_i$, $1 \leq i \leq n$, where e_i is the element whose j th coordinate is 1 if and only if $j=i$. Then $h=fg$ is the required homomorphism.

Let F be the free Heyting algebra with two free generators x, y . Then $R(F)$ is the free Boolean algebra with free generators $\neg \neg x, \neg \neg y$. This follows from the fact that the canonical map $\varphi: F \rightarrow R(F)$ is an epimorphism, and the elements $\neg \neg x \neg \neg y, \neg \neg x \neg y, \neg x \neg \neg y$ and $\neg x \neg y$ are all > 0 (as can be seen by mapping F onto the free Boolean algebra with two free generators). These four elements are the atoms of $R(F)$.

THEOREM 4.4. *The subalgebra of F generated by $\neg \neg x \neg \neg y$ and $\neg \neg x \neg y$ is infinite.*

Proof. We shall construct a Heyting algebra H containing two elements x_1, y_1 such that the subalgebra generated by $\neg \neg x_1 \neg \neg y_1$ and $\neg \neg x_1 \neg y_1$ is infinite. Let S be an infinite set which is partitioned into a triply infinite sequence of non-empty disjoint sets a_n, b_n and c_n , $0 \leq n < \omega$. That is,

$$\begin{aligned} a_i \cap b_j &= a_i \cap c_j = b_i \cap c_j = \emptyset & \text{for all } i, j, \\ a_i \cap a_j &= b_i \cap b_j = c_i \cap c_j = \emptyset & \text{for } i \neq j, \end{aligned}$$

and

$$S = \bigcup_{n < \omega} (a_n \cup b_n \cup c_n).$$

Let $x_0 = a_0, y_0 = b_0, z_0 = c_0$, and for each $n < \omega$, let

$$\begin{aligned} s_n &= \bigcup_{i < n} (a_i \cup b_i \cup c_i), & x_{n+1} &= a_{n+1} \cup b_n \cup c_n \cup s_n, \\ y_{n+1} &= a_n \cup b_{n+1} \cup c_n \cup s_n, & z_{n+1} &= a_n \cup b_n \cup c_{n+1} \cup s_n. \end{aligned}$$

We have

$$x_n \subset x_m \text{ for } m \geq n+2, \text{ and } x_n \subset y_m, x_n \subset z_m \text{ for } m \geq n+1,$$

as well as similar inequalities obtained by permuting x , y and z .

Also for all n

$$x_n \cap x_{n+1} = y_{n-1} \cup z_{n-1}, \quad \text{and} \quad x_n \cap y_n = z_{n-1} \cup z_{n-2}$$

(with the convention $x_n = y_n = z_n = \emptyset$ for $n < 0$), and similar equations obtained by permutation. Therefore we may use $\{x_n : n < \omega\} \cup \{y_n : n < \omega\} \cup \{z_n : n < \omega\}$ as a basis for a topology of S . Clearly, every infinite union of distinct basic open sets is equal to S . Let H be the algebra of all open subsets of S . It is not difficult to see that for $n \geq 0$, $x_n \rightarrow (y_n \cup z_n) = x_{n+1}$, $y_n \rightarrow (x_n \cup z_n) = y_{n+1}$, and $z_n \rightarrow (x_n \cup y_n) = z_{n+1}$. Therefore H is generated by x_0 , y_0 , and z_0 . But $\neg x_0 = x_1$, $\neg y_0 = y_1$, $\neg z_0 = z_1$, $\neg x_1 = x_0$, $\neg y_1 = y_0$, and $\neg z_1 = z_0$. Therefore $\neg \neg x_1 \neg \neg y_1 = z_0$ and $\neg \neg x_1 \neg y_1 = y_0$, and y_0 and z_0 generate H because $x_0 = \neg y_0 \neg z_0$.

THEOREM 4.5. *Suppose α is a node in a projective Heyting algebra H . If α has more than two immediate successors, then $H^{(\alpha)}$ is infinite.*

Proof. Let F be the free Heyting algebra with two free generators x , y . Let $b_1 = \neg \neg x \neg \neg y$, $b_2 = \neg \neg x \neg y$, $b_3 = \neg x \neg \neg y$, and $b_4 = \neg x \neg y$ be the atoms of $R(F)$. Let a_1, a_2, a_3 be three distinct atoms of $H^{(\alpha)}$, and let $a_4 = a_3$. By 4.3, there exists a homomorphism $h: H^{(\alpha)} \rightarrow R(F)$ such that $h(u) = \Sigma\{b_i : u \geq a_i\}$. Let $\varphi: F \rightarrow R(F)$ be the canonical epimorphism. By 1.1(8), $\varphi(u) = 0$ only for $u = 0$. Since $H = H_{(\alpha)} * H^{(\alpha)}$, 4.1 furnishes a homomorphism $g: H^{(\alpha)} \rightarrow F$ such that $\varphi g = h$. Now $g((a_1 \rightarrow \alpha) \rightarrow \alpha) = \neg \neg g(a_1) = \varphi g(a_1) = h(a_1) = b_1$, and similarly $g((a_2 \rightarrow \alpha) \rightarrow \alpha) = b_2$. Therefore $g(H^{(\alpha)})$ is infinite by 4.4. Hence $H^{(\alpha)}$ is infinite.

LEMMA 4.6. *Let H be a Heyting algebra and C be the three-element chain $\{0, \beta, 1\}$. Then there exists a homomorphism $h: H \rightarrow C$ such that $F_1 = \{x \in H : h(x) \geq \beta\}$ and $F_2 = \{x \in H : h(x) = 1\}$ if and only if:*

- (1) F_1 is a maximal filter.
- (2) F_2 is a prime filter.
- (3) $F_2 \subseteq F_1$.
- (4) For all $x \in H$, if $x \notin F_1$ then $\neg x \in F_2$.
- (5) For all $x, y \in H$, if $x, y \in F_1 - F_2$, then $x \rightarrow y \in F_2$.

Proof. This is easily checked by straightforward verification.

LEMMA 4.7. *Suppose H is a finite Heyting algebra, a is an atom of H , y is an immediate successor of a , and a is the only atom $\leq y$. Let C be the three-element chain $\{0, \beta, 1\}$. Then there exists a homomorphism $h: H \rightarrow C$ such that $h(a) = \beta$.*

Proof. Let $F_1 = H^{(a)}$ and $F_2 = H^{(y)}$. Then $F_2 \subseteq F_1$ and F_1 is a maximal filter by 4.2. To prove that F_2 is a prime filter, suppose $u \not\geq y$. Then $yu = 0$ or $yu \geq$ some atom of H . Therefore since $yu < y$, either $yu = 0$ or $yu = a$. If $u \not\geq y$ and $v \not\geq y$, but $u + v \geq y$, then $y = yu + yv \leq a$, which is a contradiction. To prove condition (4) of 4.6, suppose $x \notin F_1$. Then xy cannot be $\geq$ any atom of H . Hence $xy = 0$ and so

$y \leq \neg x$. For condition (5), suppose $u, v \in F_1 - F_2$. As before, $yu = a$. Hence $yu \leq v$, which implies $y \leq u \rightarrow v$. Lemma 4.6 now yields the desired homomorphism.

THEOREM 4.8. *If α is a node in a finite projective Heyting algebra H , and α has exactly two immediate successors a_1, a_2 then $a_1 + a_2$ is a node.*

Proof. Suppose there exists an element z incomparable with $a_1 + a_2$. Since α is a node, we must have $\alpha < z$. Without loss of generality we may assume $a_1 < z$ and so $a_2 \not\leq z$. Let y be an immediate successor of a_1 such that $y \leq z$. By 4.7 there exists a homomorphism $h_1: H^{(\alpha)} \rightarrow C = \{0, \beta, 1\}$ such that $h_1(a_1) = \beta$. By 4.2 there exists a homomorphism $h_2: H^{(\alpha)} \rightarrow B_2$ such that $h_2(u) = 1$ if and only if $u \geq a_2$. Define $h: H^{(\alpha)} \rightarrow C \times B_2$ by $h(u) = (h_1(u), h_2(u))$. Let F be the free Heyting algebra with one free generator x . Let $f: F \rightarrow C \times B_2$ be the homomorphism such that $f(x) = (\beta, 0)$. Since $\neg(\beta, 0) = (0, 1)$ and $\neg(0, 1) = (1, 0)$, $(\beta, 0)$ generates $C \times B_2$. Therefore f is an epimorphism. In [5] there is a complete description of F . From it we see that the only members of F which are $\not\leq \neg x + \neg \neg x$ are $0, x, \neg x, x + \neg x, \neg \neg x$, and $\neg \neg x \rightarrow x$. Therefore $f(u) = (0, 0)$ only for $u = 0$, and $f(u) = (\beta, 0)$ only for $u = x$. Therefore by 4.1, there exists a homomorphism $g: H^{(\alpha)} \rightarrow C \times B_2$ such that $fg = h$. Since $fg(a_1) = h(a_1) = (\beta, 0)$, $g(a_1) = x$. But x generates F , and F is infinite. Hence $H^{(\alpha)}$ is infinite, which is a contradiction.

THEOREM 4.9. *If H is a projective Heyting algebra, then 1_H is join irreducible.*

Proof. The map $f: H \oplus 1 \rightarrow H$ such that $f(1) = 1_H$ and $f(u) = u$ for $u \in H$ is an epimorphism. Since H is projective, there exists a homomorphism $g: H \rightarrow H \oplus 1$ such that $fg = I_H$. Suppose $1_H = x + y$. Then $g(x) + g(y) = 1$. Since 1 is join irreducible in $H \oplus 1$, either $g(x) = 1$ or $g(y) = 1$. Therefore either $x = fg(x) = 1_H$ or $y = 1_H$.

THEOREM 4.10. *If H is a finite Heyting algebra, then H is projective if and only if H is of the form $H = H_0 * \cdots * H_n$, where $H_n \cong B_2$ and for each $i < n$, $H_i \cong B_2$ or $H_i \cong B_4$.*

Proof. The sufficiency was proved in 3.13 for $n > 0$ and B_2 is obviously projective. Suppose H is projective. Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n+1} = 1$ be the nodes of H , and let $H_i = \{x \in H : \alpha_i \leq x \leq \alpha_{i+1}\}$, $0 \leq i \leq n$. By 4.5, each node has at most two immediate successors in H . If α_i has only one immediate successor a , then a is a node and so $a = \alpha_{i+1}$. Therefore $H_i = \{\alpha_i, \alpha_{i+1}\}$, which is isomorphic with B_2 . If α_i has two immediate successors a_1, a_2 , then $\alpha_{i+1} \geq a_1 + a_2$. Since $a_1 + a_2$ is a node by 4.8, $a_1 + a_2 = \alpha_{i+1}$. Therefore $H_i = \{\alpha_i, a_1, a_2, \alpha_{i+1}\}$, which is isomorphic with B_4 . Finally 4.9 shows that H_n cannot be isomorphic with B_4 .

5. Further results.

LEMMA 5.1. *If H is a projective Heyting algebra, then so are $B_4 * H$ and $B_2 * H$.*

Proof. Let β and γ be the atoms of B_4 . Let $\{a_i : i \in I\}$ be the set of members of $H - \{0_H\}$. Let F be a free Heyting algebra with free generating set $\{x\} \cup \{x_i : i \in I\}$.

There exists a homomorphism $f: F \rightarrow B_4 * H$ such that $f(x) = \beta$ and $f(x_i) = a_i$, $i \in I$. We have $f(\neg x) = \gamma$, $f(\neg \neg x) = \beta$, and $f(\neg x + \neg \neg x) = \beta + \gamma = 0_H$. Therefore f is an epimorphism. Let $K = \{u \in F : u \geq \neg x + \neg \neg x\}$. Let f_1 be the restriction of f to K . Since $f(\neg x + \neg \neg x + x_i) = a_i$, and by 3.6 K is a Heyting algebra which is a pre-subalgebra of F , $f_1: K \rightarrow H$ is an epimorphism. By the projectivity of H , there exists a homomorphism $g_1: H \rightarrow K$ such that $f_1 g_1 = I_H$. Let $C_4 = \{0_F, \neg x, \neg \neg x, \neg x + \neg \neg x\}$. By 1.1(10), (11) and (7), C_4 is a pre-subalgebra of F . Also the map $g_0: B_4 \rightarrow C_4$ such that $g_0(0) = 0_F$, $g_0(\beta) = \neg \neg x$, $g_0(\gamma) = \neg x$ and $g_0(1) = \neg x + \neg \neg x$ is an isomorphism. Therefore, by 3.9, the map $g: B_4 * H \rightarrow C_4 * K$ defined by $g(u) = g_1(u)$ for $u \in H$, and $g(u) = g_0(u)$ for $u \in B_4$, is a homomorphism. Furthermore $C_4 * K$ is a subalgebra of F , since if $u \geq \neg x + \neg \neg x$, 1.1(5) shows that $u \rightarrow \neg \neg x = \neg \neg x$, $u \rightarrow \neg x = \neg x$ and also $u \rightarrow 0_F = 0_F$. Therefore g may be regarded as a homomorphism $g: B_4 * H \rightarrow F$. Clearly $fg(u) = u$ for all $u \in B_4 * H$. Therefore $B_4 * H$ is a retract of F , and so $B_4 * H$ is projective. The proof of the other half of the lemma is similar. Let $f: F \rightarrow B_2 * H$ be the epimorphism such that $f(x) = 0_H$ and $f(x_i) = a_i$, $i \in I$. Let $K = \{u \in F : u \geq x + \neg x\}$. The restriction f_1 of f to K is an epimorphism of K onto H . There exists a homomorphism $g_1: H \rightarrow K$ such that $f_1 g_1 = I_H$. Let $C_2 = \{0_F, x + \neg x\}$. As before we can extend g_1 to a homomorphism $g: B_2 * H \rightarrow F$ such that $fg(u) = u$ for all $u \in B_2 * H$.

THEOREM 5.2. *If H_1 is a projective Heyting algebra, and H_0 is a finite Heyting algebra such that $H_0 \oplus 1$ is projective, then $H_0 * H_1$ is projective.*

Proof. This follows from 5.1 and 4.10.

LEMMA 5.3. *Let H be a projective Heyting algebra. Suppose S is an infinite subset of H which is closed under $\rightarrow$. Then $\Sigma(S - \{1\}) = 1$.*

Proof. Suppose the conclusion is false. Then there exists $a \in H$ such that $1 > a \geq u$ for all $u \in S - \{1\}$. Let $f: F \rightarrow H$ be an epimorphism, where F is a free Heyting algebra. There exists a homomorphism $g: H \rightarrow F$ such that $fg = I_H$. Let $\alpha = g(a)$. Then $\alpha \neq 1$, since $f(\alpha) = a < 1$. Also $\alpha \geq g(u)$ for all $u \in S - \{1\}$. Since g is a monomorphism, $g(S)$ is infinite and $g(S)$ is closed under $\rightarrow$. By [4, Lemma 2.22], there exists a finite Heyting algebra K , and a homomorphism $h: F \rightarrow K$ such that $h(\alpha) \neq 1$. Since $g(S)$ is infinite and K is finite, we have $h(v) = h(w)$ for some $v, w \in g(S)$ such that $v \neq w$. If, say, $v \not\leq w$, then $v \rightarrow w \neq 1$. Hence $h(\alpha) \geq h(v \rightarrow w) = h(v) \rightarrow h(w) = 1$, which is a contradiction.

THEOREM 5.4. *Let H be a chain. Then H is a projective Heyting algebra if and only if H is either finite or has order type $\omega + 1$.*

Proof. The sufficiency follows from 3.13. Suppose H is projective. Let $a \in H$, $a \neq 1$. Then $H_{(a)} \cup \{1\}$ is closed under $\rightarrow$. By 5.3, $H_{(a)}$ is finite. Since $H_{(a)}$ is finite for all $a < 1$, the conclusion follows.

A partial converse of 5.2 is the following.

THEOREM 5.5. *Suppose H_0, H_1 are Heyting algebras such that $H_0 * H_1$ is projective. Then H_0 is finite and $H_0 \oplus 1$ is projective.*

Proof. Let $S = H_0 \cup \{1_{H_1}\}$. Then by 3.8 and 5.3, H_0 is finite. Let F be a maximal filter in H_1 . By 4.2, there exists a homomorphism $g_1: H_1 \rightarrow B_2$ such that $g(u) = 1$ if and only if $u \in F$. By 3.9, there exists a homomorphism $g: H_0 * H_1 \rightarrow H_0 * B_2$ such that $g(u) = u$ for $u \in H_0$ and $g(u) = g_1(u)$ for $u \in H_1$. Therefore $H_0 * B_2 = H_0 \oplus 1$ is a retract of $H_0 * H_1$, and so $H_0 \oplus 1$ is projective.

We have not been able to decide whether the projectivity of $H_0 * H_1$ implies that of H_1 .

REFERENCES

1. G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ., vol. 25, Amer. Math. Soc., Providence, R. I., 1967. MR 37 #2638.
2. H. B. Curry, *Foundations of mathematical logic*, McGraw-Hill, New York, 1963. MR 26 #6036.
3. P. R. Halmos, *Lectures on Boolean algebras*, Van Nostrand Math. Studies, no. 1, Van Nostrand, Princeton, N. J., 1963. MR 29 #4713.
4. J. C. C. McKinsey and A. Tarski, *On closed elements in closure algebras*, Ann. of Math. (2) 47 (1946), 122–162. MR 7, 359.
5. I. Nishimura, *On formulas of one variable in intuitionistic propositional calculus*, J. Symbolic Logic 25 (1960), 327–331. MR 26 #25.
6. H. Rasiowa and R. Sikorski, *The mathematics of metamathematics*, Monogr. Mat., Tom 41, PWN, Warsaw, 1963. MR 29 #1149.

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